Aspects of Quantum Cosmology

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09MS002

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- The very early universe before the Planck Time scale when the length of the universe is order of Planck length.

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- The very early universe before the Planck Time scale when the length of the universe is order of Planck length.
- The thermodynamical arrow of time and its matching with cosmological arrow of time.

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- When we study the evolution of universe time is alreday a coordinate
- We need to find a parameter which is timelike against which our universe will evolve.
- We need to build up a configuration space for universe where we will define Quantum Dynamics.
- **The definition of Probability is not so well formulated.Some anisotropic models reveal time dependent norm when we define probability in standard way**
- **Our aim is to look for a suitable definition of probablity so as to alleviate this time dependence leading to Unitarity of evolution operator**

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Define extrinsic curvature of the hypersurface to be

$$K_{ij} = -n_{i;j} = \frac{1}{2N}\left(N_{i|j} + N_{j|i} - \frac{\partial h_{ij}}{\partial t}\right) \tag{2}$$

where n_i is normal to the spacelike hypersurface.

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where n_i is normal to the spacelike hypersurface.

$$S = \frac{1}{4\kappa^2}\left[\int_{\mathcal{M}} d^4x\sqrt{-g}R_{(4)} + 2\int_{\partial\mathcal{M}} d^3x\sqrt{h}K\right] + S_{matter} \tag{3}$$

Hamiltonian formulation

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$$\pi^0 \equiv \frac{\partial L}{\partial \dot{N}} = 0 \tag{6}$$

$$\pi^i \equiv \frac{\partial L}{\partial \dot{N}_i} = 0 \tag{7}$$

Writing Down the Hamiltonian

$$H = \int d^3x \left(\pi^0 \dot{N} + \pi^i \dot{N}_i + N\mathcal{H} + N_i \mathcal{H}^i \right) \tag{8}$$

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$$\mathcal{H} = 0 \tag{11}$$

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Introducing DeWitt Metric

$$\mathcal{H} = 4\kappa^2 \mathcal{G}_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}}{4\kappa^2}R_{(3)} - \sqrt{h}T^0_0 \tag{13}$$

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- If we consider a phase space described by h_{ij} and π^{ij}; $\mathcal{G}_{ijkl}$ will provide a metric on that space

Superspace

Consider the space of all Riemannian 3-metrics and matter configuration on spatial hypersurfaces, Σ,

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Hence **Superspace** is defined to be

$$Superspace(\Sigma) \equiv \frac{Riem(\Sigma)}{Diff(\Sigma)} \tag{16}$$

Metrizing the Superspace

The DeWitt metric provides a metric on Superspace, given by

$$\mathcal{G}_{AB}(x) = \mathcal{G}_{(ij)(kl)}(x) \tag{17}$$

where $A, B \in \{h_{ij} : i \leq j, i, j \in \{1, 2, 3\}\}$

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Superspace for Homogeneous Closed Universe

$$S = \int dt \left[\frac{1}{2N} \mathcal{G}_{AB} \dot{q}^A \dot{q}^B - NU(q) \right] \tag{20}$$

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$$H = N \left[\frac{1}{2} \mathcal{G}^{AB} \pi_A \pi_B + U(q) \right] \quad (21)$$

$$\left[\frac{1}{2} \mathcal{G}^{AB} \pi_A \pi_B + U(q) \right] = 0 \quad (22)$$

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The conserved current is

$$j^\alpha = \frac{-\imath\hbar}{2}\mathcal{G}^{\alpha\beta}\left(\psi^*\nabla_\beta\psi - \psi\nabla_\beta\psi^*\right) \tag{25}$$

Probability

We take ansatz $\psi = A(q)e^{\imath\frac{S(q)}{\hbar}}$ to arrive

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Hence probability is defined to be

$$P = \int j^\alpha d\Sigma_\alpha \tag{27}$$

Small Quantum Subsystem

Consider a small quantum subsystem with $H_{q'}$ being the hamiltonian:

$$ds^2 = \mathcal{G}_{\alpha\beta} q^\alpha q^\beta + \mathcal{G}_{\alpha m} q^\alpha q'^m + \mathcal{G}_{mn} q'^m q'^n \quad (28)$$

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$$\mathcal{G}_{\alpha m}(q, q') = O(\lambda) \tag{30}$$

Arriving the Schrodinger Equation for Subsystem

$$\left[\frac{\hbar^2}{2}\nabla^2 - U - H_{q'}\right]\Psi = 0 \tag{31}$$

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$$NH_{q'}\chi = \imath\hbar\frac{\partial\chi}{\partial t} \qquad (32)$$

Conserved Currents

- For the universe $j^{\alpha} = |A|^2 \hbar \nabla^{\alpha} S |\chi|^2 = j_0^{\alpha} |\chi|^2$

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Normalisation condition yields

$$\int j_0^{\alpha} d\Sigma_{\alpha} \int \rho_{\chi} d\Omega_{q'} = 1 \tag{33}$$

- This sets normalisation conditon on the probability of small quantum subsytem.

Probability conservation in Subsystem

$$\nabla_\alpha j^\alpha + \nabla_m j^m = 0 \tag{34}$$

- Write $j^\alpha = (|A|^2 \hbar \nabla^\alpha S)\rho_\chi$ and use $\nabla_\alpha j_0^\alpha = 0$

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$$\frac{1}{N}\frac{\partial \rho_\chi}{\partial t} + \partial_m j_\chi^m = 0 \tag{35}$$

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- Everything falls into place and we get back our standard QM result!!

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- The DeWitt metric always has a signature of $(-, +, +..)$. So we will try to take coordinate with opposite signature as time.
- In that case the resulting equation may not be first order in time derivative unlike Schrodinger equation.

References

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THANK YOU